CSE 150A-250A AI: Probabilistic Models

Lecture 8

Fall 2025

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Slides adapted from previous versions of the course (Prof. Lawrence, Prof. Alvarado, Prof Berg-Kirkpatrick)

Agenda

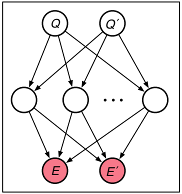
Review

Learning in BNs

Markov models

Naive Bayes models

Review



- **Initialization**

  Fix evidence nodes to observed values e, e'.

  Initialize non-evidence nodes to random values.

- **Repeat N times**

  Pick a non-evidence node X at random.

  Use **Bayes rule** to compute $P(X|B_X)$.

  Resample $x \sim P(X|B_X)$.

  Take a snapshot of all the nodes in the BN.

- **Estimate**

  Count the snapshots $N(q,q') \leq N$ with $Q{=}q$ and $Q'{=}q'$.

$$P(Q{=}q, Q'{=}q'|E{=}e, E'{=}e') \approx \frac{N(q,q')}{N}$$

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time to stationary dist.

3. Theoretical guarantees for mixing time, in practice we use burn in time.
4. The estimates from MCMC converge in the limit:

$$\lim_{N \to \infty} \frac{N(q, q')}{N} \rightarrow P(Q = q, Q' = q' | E = e, E' = e')$$

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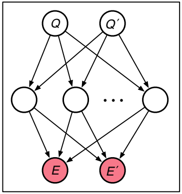
  MCMC can be much faster in this situation.



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  Visual object recognition
  Recommender systems

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  The data may be unrepresentative or too limited.
  This is one failure mode of ML estimation.

Learning with complete data and tabular CPTs

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Learning with complete data and tabular CPTs

ASSUMPTIONS

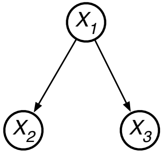
1. The DAG is fixed (and known) over a finite set of discrete random variables $\{X_1, X_2, \ldots, X_n\}$.

2. The data consists of T complete (or fully observed) instantiations of all the nodes in the BN.

3. CPTs enumerate $P(X_i = x | \mathrm{pa}(X_i) = \pi)$ as lookup tables; each must be estimated for all values of x and π.

Example

- Fixed DAG over discrete random variables



$$
\begin{aligned}
X_1 &\in \{1, 2, 3\} \\
X_2 &\in \{1, 2, 3, 4\} \\
X_3 &\in \{1, 2, 3, 4, 5\}
\end{aligned}
$$

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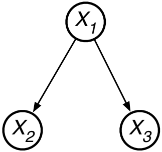

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- Data set

example	x_1	x_2	x_3
1	1	4	5
2	3	2	4
3	2	1	3
⋮	⋮	⋮	⋮
T	1	3	2

→ 1 data poin
⋮
T

Example

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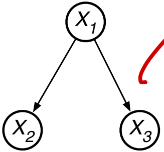

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Note that if T is sufficiently large, some rows are destined to repeat.

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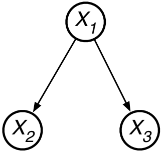

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We can also denote the data set as $\left\{\left(x_1^{(t)}, x_2^{(t)}, x_3^{(t)}\right)\right\}_{t=1}^{T}$.

Example

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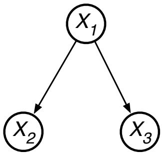

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How to choose the CPTs so that the BN maximizes the probability of this data set?

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- **Probability of IID data**

$$P(data) = \prod_{t=1}^{T} P\left(X_1 = x_1^{(t)}, X_2 = x_2^{(t)}, \ldots, X_n = x_n^{(t)}\right)$$

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$$= \prod_{i=1}^{n} P\left(X_i = x_i^{(t)} \middle| X_1 = x_1^{(t)}, \ldots, X_{i-1} = x_{i-1}^{(t)}\right) \quad \boxed{\text{product rule}}$$

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all rows x

- Probability of t^{th} example

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\begin{aligned}
& P\left(X_1 = x_1^{(t)}, X_2 = x_2^{(t)}, \ldots, X_n = x_n^{(t)}\right) \\
&= \prod_{i=1}^{n} P\left(X_i = x_i^{(t)} \,\middle|\, X_1 = x_1^{(t)}, \ldots, X_{i-1} = x_{i-1}^{(t)}\right) \quad \boxed{\text{product rule}} \\
&= \prod_{i=1}^{n} P\left(X_i = x_i^{(t)} \,\middle|\, \mathrm{pa}(X_i) = \mathrm{pa}_i^{(t)}\right) \quad \boxed{\text{conditional independence}}
\end{aligned}
$$

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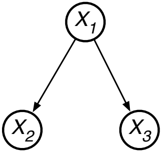

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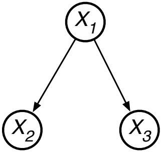

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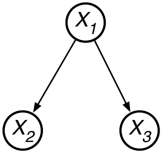

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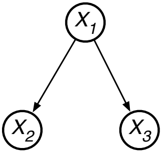

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1	3	2

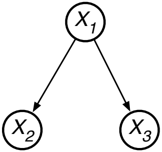
$$\begin{aligned}\mathrm{count}(X_1 = 1) &= 3\\ \mathrm{count}(X_1 = 2) &= 2\\ \mathrm{count}(X_1 = 3) &= 1\end{aligned}$$

Counting co-occurrences

- Counts

  Let $\mathrm{count}(X_i = x, \mathrm{pa}_i = \pi)$ denote the number of examples where $X_i = x$ and $\mathrm{pa}_i = \pi$.

- Example



X_1	X_2	X_3
1	4	5
3	2	4
2	1	3
2	1	4
1	3	5
1	3	2

$$\mathrm{count}(X_1 = 1) = 3$$
$$\mathrm{count}(X_1 = 2) = 2$$
$$\mathrm{count}(X_1 = 3) = 1$$
$$\mathrm{count}(X_2 = 1, X_1 = 2)$$

Counting co-occurrences

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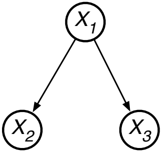

X_1	X_2	X_3
1	4	5
3	2	4
2	1	3
2	1	4
1	3	5
1	3	2

$$\begin{aligned} \mathrm{count}(X_1 = 1) &= 3 \\ \mathrm{count}(X_1 = 2) &= 2 \\ \mathrm{count}(X_1 = 3) &= 1 \\ \mathrm{count}(X_2 = 1, X_1 = 2) &= 2 \end{aligned}$$

Counting co-occurrences

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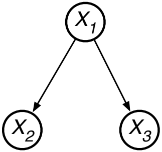

x_1	x_2	x_3
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$$
\begin{aligned}
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\mathrm{count}(X_1 = 3) &= 1 \\
\mathrm{count}(X_2 = 1, X_1 = 2) &= 2 \\
\mathrm{count}(X_2 = 3, X_1 = 1) &= 2
\end{aligned}
$$

Counting co-occurrences

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  Let $\mathrm{count}(X_i = x, \mathrm{pa}_i = \pi)$ denote the number of examples where $X_i = x$ and $\mathrm{pa}_i = \pi$.

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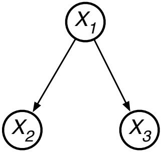

x_1	x_2	x_3
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3	2	4
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2	1	4
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\mathrm{count}(X_1 = 1) &= 3 \\
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\mathrm{count}(X_1 = 3) &= 1 \\
\mathrm{count}(X_2 = 1, X_1 = 2) &= 2 \\
\mathrm{count}(X_2 = 3, X_1 = 1) &= 2 \\
&\vdots \\
\mathrm{count}(X_3 = 5, X_1 = 1) &= 2
\end{aligned}
$$

Counting co-occurrences

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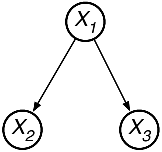

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$$
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\mathrm{count}(X_2=1, X_1=2) &= 2\\
\mathrm{count}(X_2=3, X_1=1) &= 2\\
&\vdots\\
\mathrm{count}(X_3=5, X_1=1) &= 2
\end{aligned}
$$

Note: these counts can be compiled in one pass through the data set.

Computing the log-likelihood

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Next: replace the unweighted sum over examples at each node by a weighted sum over its values and those of its parents.

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$$= \sum_{i=1}^{n} \sum_{x} \sum_{\pi} \mathrm{count}(X_i = x, \mathrm{pa}_i = \pi) \log P(X_i = x | \mathrm{pa}_i = \pi)$$

weighted

Computing the log-likelihood

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a + a + a

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3a

$$= \sum_{i=1}^{n} \sum_{x} \sum_{\pi} \mathrm{count}(X_i = x, \mathrm{pa}_i = \pi) \log P(X_i = x | \mathrm{pa}_i = \pi)$$

weighted

A - got it
B → Huh!

These two expressions compute the exact same sum!

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$$\boxed{\text{weighted}}$$

These two expressions compute the exact same sum!

But the latter has a much more appealing form …

Interpreting the log-likelihood

Interpreting the log-likelihood

$$\mathcal{L} = \sum_i \sum_x \sum_\pi \overbrace{\mathrm{count}(X_i = x, \mathrm{pa}_i = \pi)}^{\text{constants of the data}} \underbrace{\log P(X_i = x | \mathrm{pa}_i = \pi)}_{\text{CPTs to optimize}}$$

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  i — the nodes in the BN

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- The log-likelihood for complete data is a triple sum over

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- How to optimize?

  Intuitively, the larger the $\mathrm{count}(X_i{=}x, \mathrm{pa}_i{=}\pi)$,
  the larger we should choose $P(X_i{=}x|\mathrm{pa}_i{=}\pi)$.

Decomposing the log-likelihood

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- Contribution from row π of i^{th} node's CPT

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  The overall optimization over $\mathcal{L}$ reduces to many simpler and smaller optimizations over each $\mathcal{L}_{i\pi}$.

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  The overall optimization over $\mathcal{L}$ reduces to many simpler and smaller optimizations over each $\mathcal{L}_{i\pi}$.

This is a special property of ML estimation for **complete** *data.*

ML Estimation

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- Problem

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  For each node X_i in the BN,

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  For each node X_i in the BN, and for each row π of its CPT,

ML Estimation

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  For each node X_i in the BN, and for each row π of its CPT, our goal is to maximize

$$\mathcal{L}_{i\pi} \;=\; \sum_{x} \mathrm{count}(X_i\!=\!x, \mathrm{pa}_i\!=\!\pi) \log P(X_i\!=\!x|\mathrm{pa}_i\!=\!\pi)$$

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- Shorthand

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$$\begin{aligned} C_\alpha &= \mathrm{count}(X_i\!=\!\alpha, \mathrm{pa}_i\!=\!\pi) \\ p_\alpha &= P(X_i\!=\!\alpha|\mathrm{pa}_i\!=\!\pi) \end{aligned}$$

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- **Shorthand**

$$C_\alpha = \mathrm{count}(X_i = \alpha, \mathrm{pa}_i = \pi)$$
$$p_\alpha = P(X_i = \alpha | \mathrm{pa}_i = \pi)$$

$\Longrightarrow$

How to maximize $\sum_\alpha C_\alpha \log p_\alpha$ such that $\sum_\alpha p_\alpha = 1$ and $p_\alpha \geq 0$?

Maximizing the likelihood

- Compute the normalized counts:

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  Define $q_\alpha = \frac{c_\alpha}{\sum_\beta c_\beta}$ so that $\sum_\alpha q_\alpha = 1$.

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  Maximize $\left(-\sum_\alpha C_\alpha \log p_\alpha\right)$ such that $\sum_\alpha p_\alpha = 1, p_\alpha \geq 0.$

  Minimize $\sum_\alpha C_\alpha \log \frac{1}{p_\alpha}$ such that $\sum_\alpha p_\alpha = 1, p_\alpha \geq 0.$

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Minimize	$\sum_\alpha C_\alpha \log \frac{C_\alpha}{p_\alpha}$	such that	$\sum_\alpha p_\alpha = 1,\ p_\alpha \geq 0.$
Minimize	$\sum_\alpha q_\alpha \log \frac{q_\alpha}{p_\alpha}$	such that	$\sum_\alpha p_\alpha = 1,\ p_\alpha \geq 0.$

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Minimize	$\underbrace{\sum_\alpha q_\alpha \log \frac{q_\alpha}{p_\alpha}}_{\mathrm{KL}(q,p)}$	such that	$\sum_\alpha p_\alpha = 1,\ p_\alpha \geq 0.$

$\leftarrow$ KL distance

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Solution: $p_\alpha = q_\alpha$

ML solution from normalized counts

ML solution from normalized counts

$$P_{\mathrm{ML}}(X_i = x | \mathrm{pa}_i = \pi) = \frac{\mathrm{count}(X_i = x, \mathrm{pa}_i = \pi)}{\sum_{x'} \mathrm{count}(X_i = x', \mathrm{pa}_i = \pi)}$$

ML solution from normalized counts

$$P_{\mathrm{ML}}(X_i = x | \mathrm{pa}_i = \pi) = \frac{\mathrm{count}(X_i = x, \mathrm{pa}_i = \pi)}{\sum_{x'} \mathrm{count}(X_i = x', \mathrm{pa}_i = \pi)}$$

- For nodes with parents:

ML solution from normalized counts

$$P_{\mathrm{ML}}(X_i\!=\!x|\mathrm{pa}_i\!=\!\pi) \;=\; \frac{\mathrm{count}(X_i\!=\!x, \mathrm{pa}_i\!=\!\pi)}{\sum_{x'} \mathrm{count}(X_i\!=\!x', \mathrm{pa}_i\!=\!\pi)}$$

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$$P_{\mathrm{ML}}(X_i\!=\!x|\mathrm{pa}_i\!=\!\pi) \;=$$

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ML solution from normalized counts

$$P_{\mathrm{ML}}(X_i = x | \mathrm{pa}_i = \pi) = \frac{\mathrm{count}(X_i = x, \mathrm{pa}_i = \pi)}{\sum_{x'} \mathrm{count}(X_i = x', \mathrm{pa}_i = \pi)}$$

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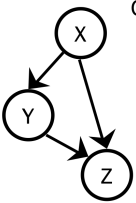
- For root nodes:

$$P_{\mathrm{ML}}(X_i = x) = \frac{\mathrm{count}(X_i = x)}{T}$$

ML Example

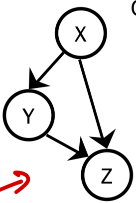


X, Y and Z are Boolean variables

Observed data:

X	Y	Z
0	0	1
0	1	0
0	1	1
0	1	0
1	0	0
1	0	0
0	1	1
1	0	0
0	1	1
0	0	1
0	1	1
1	0	0

ML Example



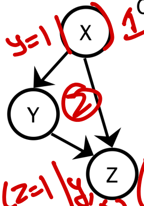
X, Y and Z are Boolean variables

Observed data:

X	Y	Z
0	0	1
0	1	0
0	1	1
0	1	0
1	0	0
1	0	0
0	1	1
1	0	0
0	1	1
0	0	1
0	1	1
1	0	0

Q. Which of the following is a parameter we would like to estimate?

A. $P(X=1)$
B. $P(Y=1)$
C. $P(X=1|Y=1)$
D. More than one of these
E. None of these



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0	1	1
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Q. Not including complements (e.g. P(X=1) and P(X=0)), how many different parameters are there to estimate?

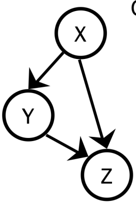
A. 3

B. 4

C. 5

D. 7

E. More than 7



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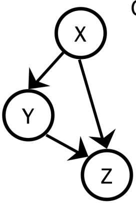
Q. What is the ML estimate for $P(Z=1|X=0, Y=0)$?

A. 0
B. 1/6
C. 1/2
D. 1
E. None of the above

ML Example



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0	1	1
0	0	1
0	1	1
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Q. Which parameter has an undefined ML estimate?

A. P(X=1)
B. P(Y=1|X=0)
C. P(Z=1|X=0, Y=0)
D. P(Z=1|X=1, Y=1)
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Properties of ML solution

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- Asymptotically correct:

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- **Asymptotically correct:**

  The more data you have, the better your estimates.
  If $P(x_1, x_2, \ldots, x_n) > 0$, then

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Markov models

Statistical language modeling

Let w_ℓ denote the ℓ^{th} word in a sentence (or text).

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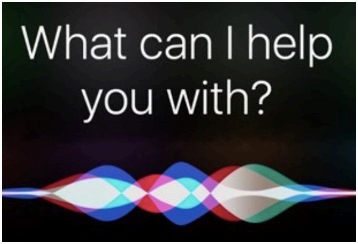
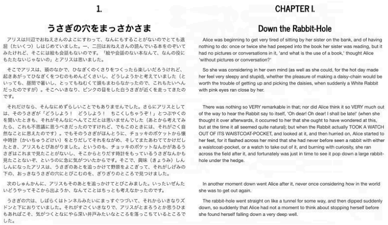


automatic speech recognition



1.

うさぎの穴をまっさかさま

アリスは川辺でおねえさんのよこにすわって、なんにもすることがないのでとても退屈（たいくつ）しはじめていました。一、二回はおねえさんの読んでいる本をのぞいてみたけれど、そこには絵も会話もないのです。「絵や会話のない本なんて、なんの役にもたたないじゃないの」とアリスは思いました。

そこでアリスは、頭のなかで、ひなぎくのくさりをつくったら楽しいだろうけれど、起きあがってひなぎくをつむのもめんどくさいし、どうしようかと考えていました（といっても、昼間で暑いし、とってもねむくて頭もまわらなかったので、これもたいへんだったのですが）。そこへいきなり、ピンクの目をした白うさぎが近くを走ってきたのです。

それだけなら、そんなにめずらしいことでもありませんでした。さらにアリスとしては、そのうさぎが「どうしよう！　どうしよう！　ちこくしちゃうぞ！」とつぶやくのを聞いたときも、それがそんなにへんてこだとは思いませんでした（あとから考えてみたら、これも不思議に思うべきだったのですけれど、でもこのときには、それがごく自然なことに思えたのです）。でもそのうさぎがほんとうに、チョッキのポケットから懐中時計（かいちゅうどけい）をとりだしてそれをながめ、そしてまたあわててかけだしたとき、アリスもとびあがりました。というのも、チョッキのポケットなんかがあるうさぎはこれまで見たことがないし、そこからとりだす時計をもっているうさぎなんかも見たことないぞ、というのに急に気がついたからです。そこで、興味（きょうみ）しんしんになったアリスは、うさぎのあとを追っかけて野原をよこぎって、それがしげみの下の、おっきなうさぎの穴にとびこむのを、ぎりぎりのところで見つけました。

次のしゅんかんに、アリスもそのあとを追っかけてとびこみました。いったいぜんたいどうやってそこから出ようか、なんてことはちっとも考えなかったのです。

うさぎの穴は、しばらくはトンネルみたいにまっすぐつづいて、それからいきなりズドンと下におりていました。それがすごくいきなりで、アリスがとまろうとか思うひまもあればこそ、気がつくとなにやら深い井戸みたいなところを落っこちているところでした。

CHAPTER I.

Down the Rabbit-Hole

Alice was beginning to get very tired of sitting by her sister on the bank, and of having nothing to do: once or twice she had peeped into the book her sister was reading, but it had no pictures or conversations in it, 'and what is the use of a book,' thought Alice 'without pictures or conversation?'

So she was considering in her own mind (as well as she could, for the hot day made her feel very sleepy and stupid), whether the pleasure of making a daisy-chain would be worth the trouble of getting up and picking the daisies, when suddenly a White Rabbit with pink eyes ran close by her.

There was nothing so VERY remarkable in that; nor did Alice think it so VERY much out of the way to hear the Rabbit say to itself, 'Oh dear! Oh dear! I shall be late!' (when she thought it over afterwards, it occurred to her that she ought to have wondered at this, but at the time it all seemed quite natural); but when the Rabbit actually TOOK A WATCH OUT OF ITS WAISTCOAT-POCKET, and looked at it, and then hurried on, Alice started to her feet, for it flashed across her mind that she had never before seen a rabbit with either a waistcoat-pocket, or a watch to take out of it, and burning with curiosity, she ran across the field after it, and fortunately was just in time to see it pop down a large rabbit-hole under the hedge.

In another moment down went Alice after it, never once considering how in the world she was to get out again.

The rabbit-hole went straight on like a tunnel for some way, and then dipped suddenly down, so suddenly that Alice had not a moment to think about stopping herself before she found herself falling down a very deep well.

machine translation

Context and expectations in language

"It's hard to wreck a nice beach."

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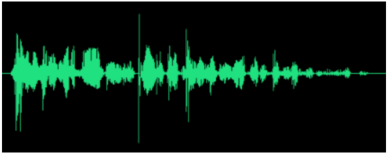

Context and expectations in language

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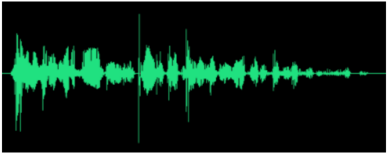

"It's hard to recognize speech."

Simplifying assumptions

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   $$=$$

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$$P(W_\ell = w' | w_{\ell-(n-1)}, \ldots, w_{\ell-1})$$

$$= \quad P(W_{s+\ell} = w' | W_{s+\ell-(n-1)} = w_{\ell-(n-1)}, \ldots, W_{s+\ell-1} = w_{\ell-1})$$

Markov models

$P(w_1, w_2, \ldots, w_L)$

Markov models

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$$= \prod_{\ell} P(w_\ell | w_1, w_2, \ldots, w_{\ell-1}) \qquad \boxed{\text{product rule}}$$

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Models of different orders

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$n = 1$

Models of different orders

$n = 1$ unigram

Markov models

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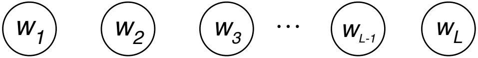

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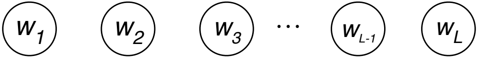

$n = 2$

Markov models

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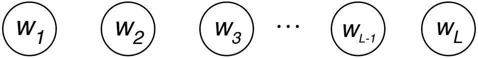

$n = 2$ bigram

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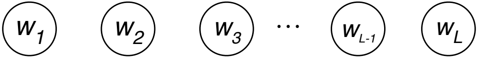

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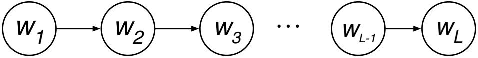

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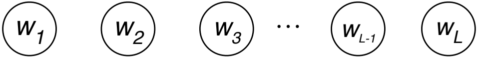
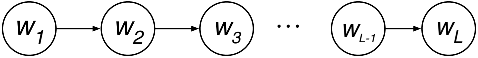


$n = 2$ bigram



$n = 3$

Markov models

Models of different orders

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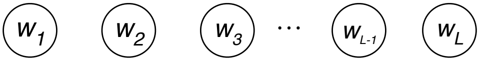
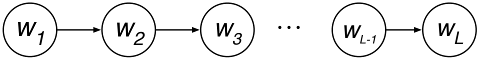


$n = 2$ bigram



$n = 3$ trigram

Models of different orders

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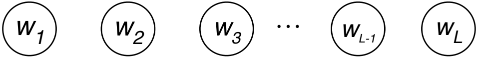

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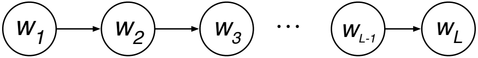

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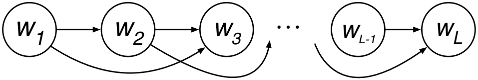
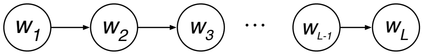
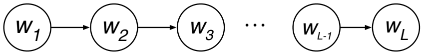


Bigram models

Bigram models



Bigram models



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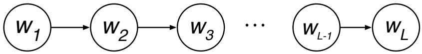
Bigram models



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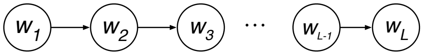
How to learn?

Bigram models



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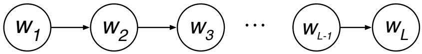
How to learn?

all unique words

Collect a large corpus of text with a well-defined vocabulary.

Bigram models



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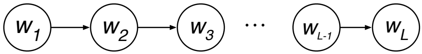
Bigram models



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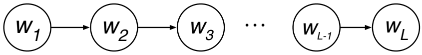
Bigram models



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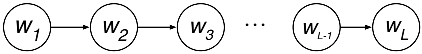

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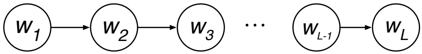
$$P_{\mathrm{ML}}(w_\ell = w' | w_{\ell-1} = w) \ =$$

Bigram models



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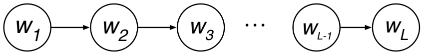
Bigram models



Note that the same CPT for $P(w_\ell = w' | w_{\ell-1} = w)$ is used at each node (for $\ell > 1$).

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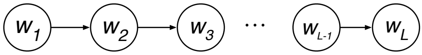

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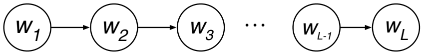

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You will explore this problem further in HW 4.

Naive Bayes models

Document classification

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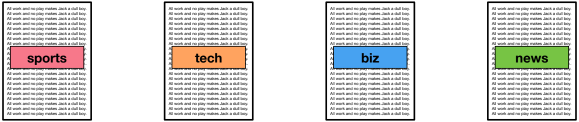

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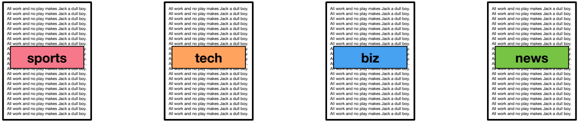

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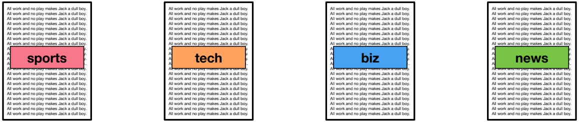
Document classification



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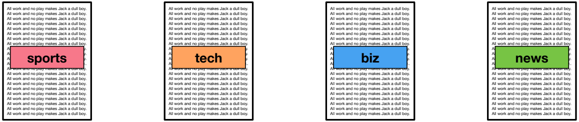

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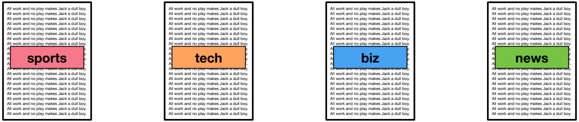

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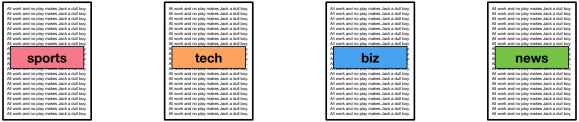

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  Let $Y \in \{1, 2, \ldots, m\}$ denote the label.

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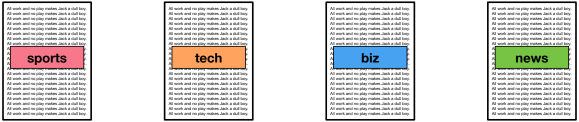

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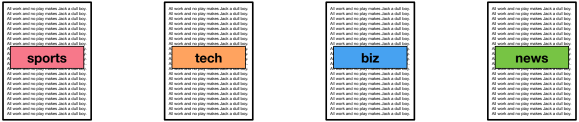

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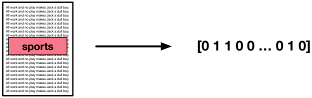
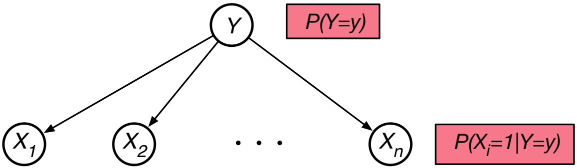
This representation maps each document to a sparse binary vector of fixed length.



Belief network

Belief network



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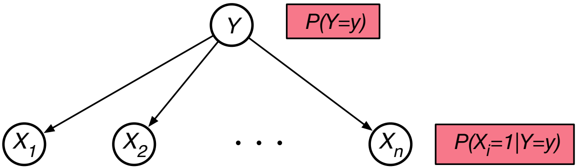
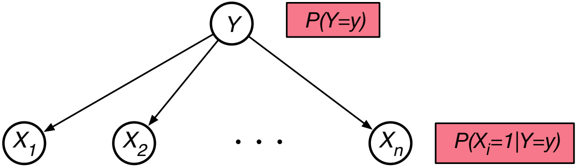


This DAG makes a fairly drastic assumption of conditional independence:

$$P(X_1, \ldots, X_n | Y) = \prod_{i=1}^{n} P(X_i | Y)$$

Belief network



This DAG makes a fairly drastic assumption of conditional independence:

$$P(X_1, \ldots, X_n | Y) = \prod_{i=1}^{n} P(X_i | Y)$$

For this reason it is called a Naive Bayes model.

Naive Bayes model

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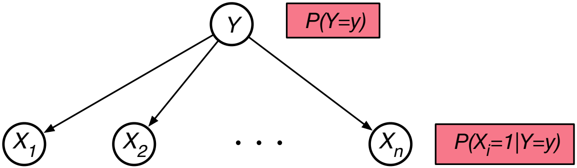

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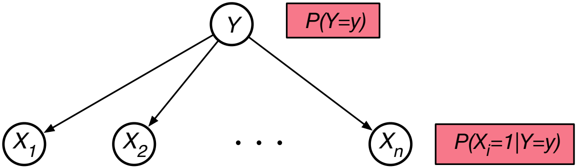
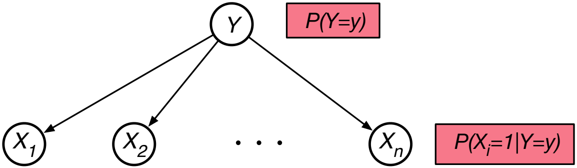


Suppose this DAG is given, but the CPTs are not specified.

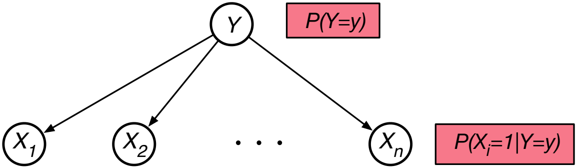
Naive Bayes model



Suppose this DAG is given, but the CPTs are not specified.

How to learn the CPTs from data?

Naive Bayes model



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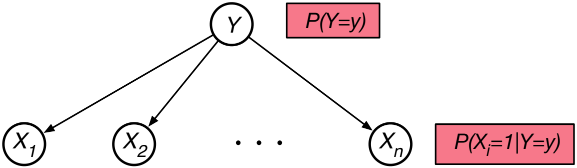


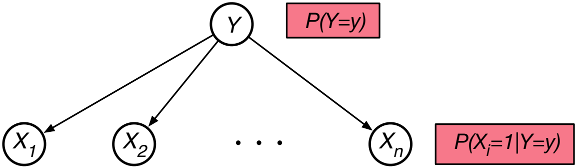
Suppose this DAG is given, but the CPTs are not specified.

How to learn the CPTs from data?

- **Collect** a large corpus of documents.
- **Label** each document by a topic.

→ supervised learning

Naive Bayes model



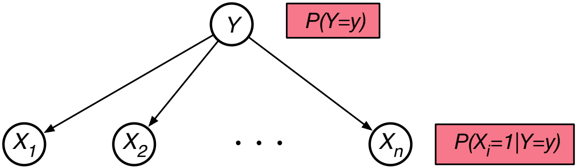
Suppose this DAG is given, but the CPTs are not specified.

How to learn the CPTs from data?

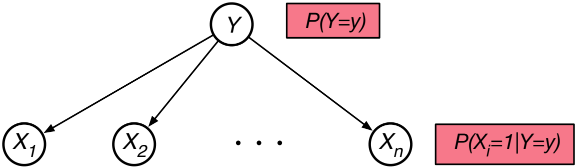
- **Collect** a large corpus of documents.
- **Label** each document by a topic.
- **Estimate** the CPTs by maximizing the likelihood.

ML estimation

ML estimation

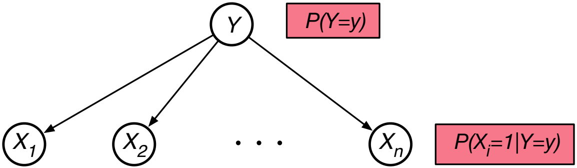


ML estimation



$P_{\mathrm{ML}}(Y=y)$

ML estimation



$$P_{\mathrm{ML}}(Y{=}y) \quad = \quad \text{fraction of documents with label } y \text{ in the corpus}$$

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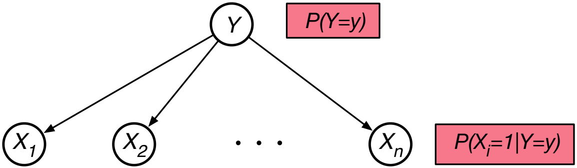
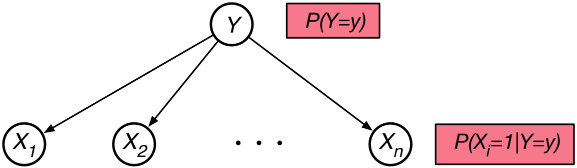


$$P_{\mathrm{ML}}(Y = y) \quad = \quad \text{fraction of documents with label } y \text{ in the corpus}$$

$$P_{\mathrm{ML}}(X_i = 1 | Y = y)$$

ML estimation



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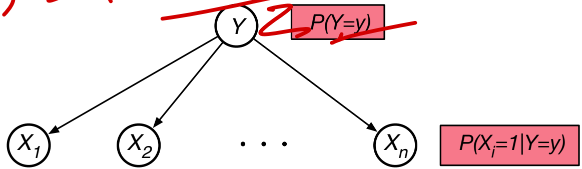
ML estimation

P(y|x) = P(x|y) P(y)



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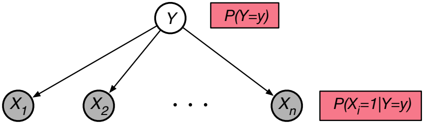
Once the model is learned, what is it good for?

predict Y | X

Inference

How to classify an unlabeled document?



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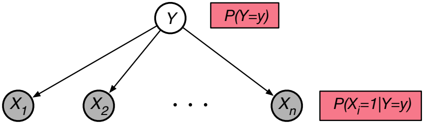

$$P(Y{=}y|X_1, X_2, \ldots, X_n)$$

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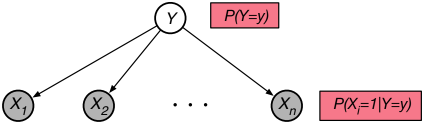

$$P(Y\!=\!y|X_1, X_2, \ldots, X_n)$$

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Bayes rule

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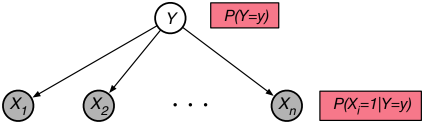

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Bayes rule

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conditional independence

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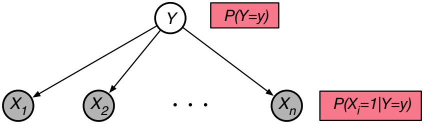

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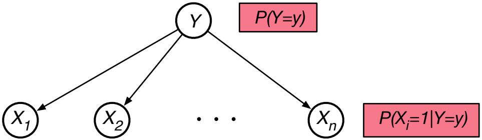
normalization

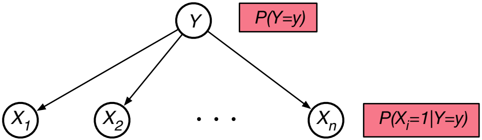
Strengths and weaknesses

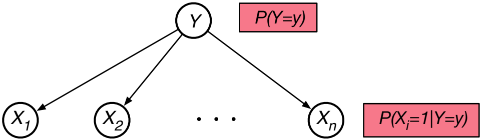
Strengths and weaknesses



Strengths and weaknesses

Strengths





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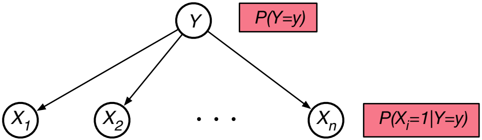
Strengths and weaknesses



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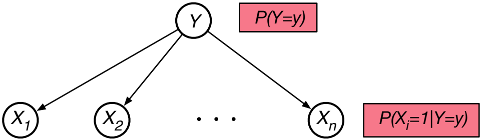

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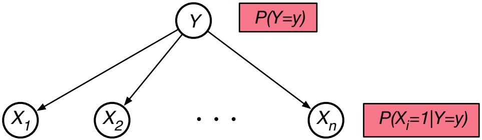

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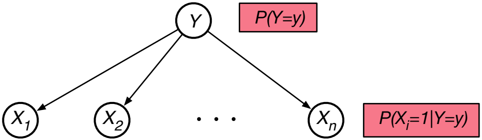

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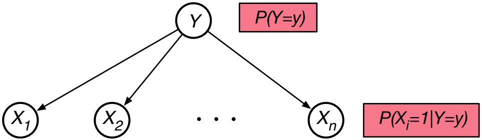
Strengths and weaknesses



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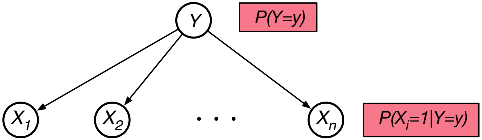

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- Etc ...

That’s all folks!