CSE 150A-250A AI: Probabilistic Methods

Lecture 3

Fall 2025

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Slides adapted from previous versions of the course (Prof. Lawrence, Prof. Alvarado, Prof Berg-Kirkpatrick)

Agenda

Review

Joint Distributions and Inference

Alarm Example

Belief networks

Review

Review

- Types of probabilities:

$P(X, Y)$	joint
$P(Y \mid X)$	conditional
$P(X)$	unconditional (or marginal)

Review

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$P(X, Y)$	joint
$P(Y\|X)$	conditional
$P(X)$	unconditional (or marginal)

- Useful rules:

$$P(A, B, C, \ldots) = P(A)\, P(B|A)\, P(C|A, B) \ldots \quad \text{product rule}$$

$$P(X|Y) = P(Y|X)P(X)/P(Y) \quad \text{Bayes rule}$$

$$P(X) = \sum_y P(X, Y = y) \quad \text{marginalization}$$

Review

- **Types of probabilities:**

$P(X, Y)$	joint
$P(Y \mid X)$	conditional
$P(X)$	unconditional (or marginal)

- **Useful rules:**

$$P(A, B, C, \ldots) = P(A)\, P(B \mid A)\, P(C \mid A, B) \ldots \quad \textbf{product rule}$$

$$P(X \mid Y) = P(Y \mid X) P(X) / P(Y) \quad \textbf{Bayes rule}$$

$$P(X) = \sum_y P(X, Y = y) \quad \textbf{marginalization}$$

- **Conditioning on background evidence E:**

$$P(A, B, C, \ldots \mid E) = P(A \mid E)\, P(B \mid A, E)\, P(C \mid A, B, E) \ldots$$

$$P(X \mid Y, E) = P(Y \mid X, E) P(X \mid E) / P(Y \mid E)$$

$$P(X \mid E) = \sum_y P(X, Y = y \mid E)$$

Marginal and conditional independence

- Marginal independence

$$\left.\begin{aligned} P(X|Y) &= P(X) \\ P(Y|X) &= P(Y) \\ P(X,Y) &= P(X)\,P(Y) \end{aligned}\right\}\ \textit{Each of these implies the other two.}$$

Marginal and conditional independence

- Marginal independence

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\left.\begin{aligned}
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P(X, Y) &= P(X)\,P(Y)
\end{aligned}\right\}
$$

Each of these implies the other two.

- Conditional independence

$$
\left.\begin{aligned}
P(X|Y, E) &= P(X|E) \\
P(Y|X, E) &= P(Y|E) \\
P(X, Y|E) &= P(X|E)\,P(Y|E)
\end{aligned}\right\}
$$

Each of these also implies the other two.

Example of conditional dependence

  

- B and E are marginally independent:

$$
\begin{aligned}
P(B) &= P(B|E) \\
P(E) &= P(E|B) \\
P(B,E) &= P(B)\,P(E)
\end{aligned}
$$

Example of conditional dependence



- B and E are marginally independent:

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\begin{aligned}
P(B) &= P(B|E) \\
P(E) &= P(E|B) \\
P(B,E) &= P(B)\,P(E)
\end{aligned}
$$

- But B and E are conditionally dependent given A:

$$
\begin{aligned}
P(B|A) &\neq P(B|E,A) \\
P(E|A) &\neq P(E|B,A) \\
P(B,E|A) &\neq P(B|A)\,P(E|A)
\end{aligned}
$$

Joint Distributions and Inference

Why Joints Matter

The joint distribution $P(X_1, \ldots, X_n)$ is a **complete description of uncertainty**.

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From the full joint distribution $P(X_1, \ldots, X_n)$, what kinds of probabilities can we compute?

A. Only marginals, $P(X_i)$

B. Only conditionals, $P(X_i | X_j)$

C. Any marginal or conditional over the variables

D. None of the above

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B. Only conditionals, $P(X_i|X_j)$
C. Any marginal or conditional over the variables
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Inference: Compute the **posterior** distribution of query variables given observed evidence.

Motivation

- Model complexity

  Suppose $X_i \in \{0, 1\}$ are binary random variables.

- **Model complexity**

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How many numbers do we need to specify the joint distribution of $P(X_1 = x_1, \ldots, X_n = x_n)$?

A. $O(n)$

B. $O(2^n)$

C. $O(n^2)$

D. $O(logn)$

$P(x_1 = 1, x_2 = 1)$

$P(x_1 = 0, x_2 = 1)$

$P(x_1 = 1, x_2 = 0)$

$= 0 \quad = 0$

- **Model complexity**

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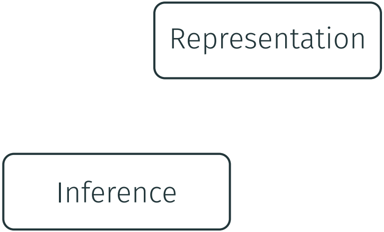
It requires $O(2^n)$ numbers to specify the joint distribution $P(X_1 = x_1, \ldots, X_n = x_n)$.

Conceptual and Practical Goals

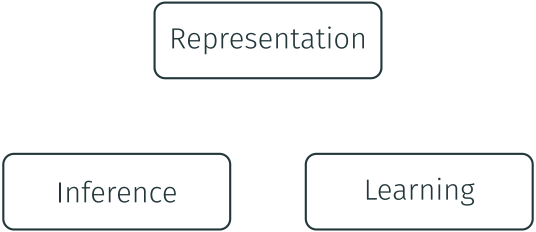
Representation

- **Representation**: compactly encode the joint.



- **Representation**: compactly encode the joint.
- **Inference**: answer queries given evidence.

Conceptual and Practical Goals

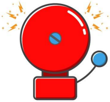


- **Representation**: compactly encode the joint.
- **Inference**: answer queries given evidence.
- **Learning**: estimate structure/parameters from data.

Alarm Example

Alarm example






- Binary random variables

$B \in \{0,1\}$	Was there a burglary?
$E \in \{0,1\}$	Was there an earthquake?
$A \in \{0,1\}$	Was the alarm triggered?
$J \in \{0,1\}$	Did Jamal call?
$M \in \{0,1\}$	Did Maya call?

Joint distribution

- Product rule

  $P(B, E, A, J, M)$

Joint distribution

- Product rule

$$P(B, E, A, J, M)$$

$$= \quad P(B)$$

Joint distribution

- Product rule

$$
\begin{aligned}
& P(B, E, A, J, M) \\
& \quad = \quad P(B)\, P(E|B)
\end{aligned}
$$

Joint distribution

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$$P(B, E, A, J, M)$$

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Joint distribution

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$$= \quad P(B)\, P(E|B)\, P(A|B, E)\, P(J|B, E, A)$$

Joint distribution

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Joint distribution

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$$P(B, E, A, J, M)$$
$$= \quad P(B)\, P(E|B)\, P(A|B, E)\, P(J|B, E, A)\, P(M|B, E, A, J)$$

Note: the above is true no matter what the variables signify.

Joint distribution

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- Domain-specific assumptions

Joint distribution

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- Domain-specific assumptions

$$P(E|B) \quad = \quad P(E)$$ marginal independence

Joint distribution

- Product rule

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$$

Note: the above is true no matter what the variables signify.

- Domain-specific assumptions

$$
\begin{aligned}
P(E|B) &= P(E) && \text{marginal independence} \\
P(J|B, E, A) &= P(J|A) && \text{conditional independence}
\end{aligned}
$$

Joint distribution

- Product rule

$$P(B, E, A, J, M)$$
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- Domain-specific assumptions

$$\begin{aligned} P(E|B) &= P(E) && \text{marginal independence} \\ P(J|B, E, A) &= P(J|A) && \text{conditional independence} \\ P(M|B, E, A, J) &= P(M|A) && \text{conditional independence} \end{aligned}$$

Completing the model

- Joint distribution

$$P(B, E, A, J, M)$$

$$= P(B)\, P(E|B)\, P(A|B, E)\, P(J|B, E, A)\, P(M|B, E, A, J)$$

Completing the model

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\begin{aligned}
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- Conditional probability tables (CPTs)

Completing the model

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- Conditional probability tables (CPTs)

$P(B=1) = 0.001$

$P(E=1) = 0.002$

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1	0	0.94
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Completing the model

- Joint distribution

$J \perp B \mid A$

$J \perp E \mid A$

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\begin{aligned}
& P(B, E, A, J, M) \\
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Inference

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&\quad = (0.001)(1-0.002)(0.94)(0.9)(0.7)
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- Any inference can be expressed in terms of joint probabilities:

$$P(B{=}1, E{=}0|M{=}1)$$

- Joint probabilities are easy to compute:

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- Any inference can be expressed in terms of joint probabilities:

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&\quad = \frac{\sum_{a,j} P(B{=}1, E{=}0, A{=}a, J{=}j, M{=}1)}{\sum_{b',e',a',j'} P(B{=}b', E{=}e', A{=}a', J{=}j', M{=}1)} \qquad \boxed{\text{marginalization}}
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\end{aligned}
$$

But this approach can be very inefficient!

Efficient inference

How to perform inference most efficiently?

Efficient inference

How to perform inference most efficiently?

1. Visualize models as directed acyclic graphs.
2. Exploit graph structure to organize and simplify calculations.

Efficient inference

How to perform inference most efficiently?

1. Visualize models as directed acyclic graphs.
2. Exploit graph structure to organize and simplify calculations.

We'll spend today on (1) and next lecture on (2).

Visualizing the model

- Joint distribution

$$
\begin{aligned}
P(B, E, A, J, M) \\
&= P(B)\, P(E|B)\, P(A|B, E)\, P(J|B, E, A)\, P(M|B, E, A, J) \\
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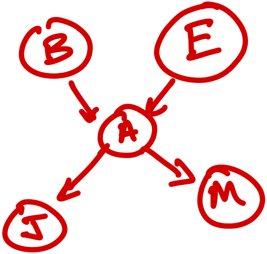
\end{aligned}
$$

- Directed acyclic graph (DAG)



- Joint distribution

$$\begin{aligned} P(B, E, A, J, M) \\ &= P(B)\,P(E|B)\,P(A|B, E)\,P(J|B, E, A)\,P(M|B, E, A, J) \\ &= P(B)\,P(E)\,P(A|B, E)\,P(J|A)\,P(M|A) \end{aligned}$$

- Directed acyclic graph (DAG)

Visualizing the model

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- Directed acyclic graph (DAG)

Absent edges encode assumptions of independence.

Visualizing the model

- Joint distribution

$$\begin{aligned} &P(B, E, A, J, M) \\ &\quad = P(B)\, P(E|B)\, P(A|B, E)\, P(J|B, E, A)\, P(M|B, E, A, J) \\ &\quad = P(B)\, P(E)\, P(A|B, E)\, P(J|A)\, P(M|A) \end{aligned}$$

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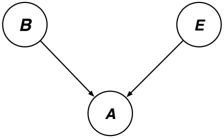

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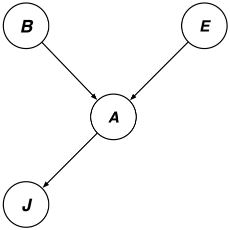

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Visualizing the model

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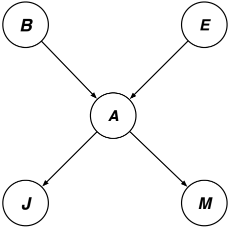

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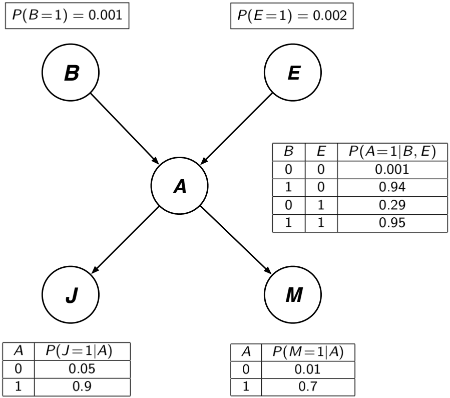

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This visual representation of the joint distribution is called a **belief network** (or a **Bayesian network**, or a **probabilistic graphical model**).

Belief networks

Definition

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A **belief network** (BN) is a directed acyclic graph (DAG) in which:

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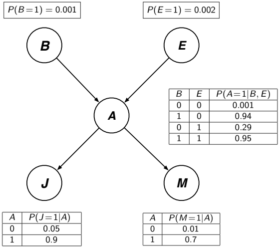
BN = DAG + CPTs



From distributions to graphs

From distributions to graphs

- It is always true from the product rule that

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- Big idea: represent conditional dependencies by a DAG.

Constructing a belief network

Three steps:

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1. Choose your random variables of interest.

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   (c) define the conditional probability table $P(X_i | \mathrm{parents}(X_i))$

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  Add the "root causes," then the variables they influence, then the next variables that are influenced, etc.

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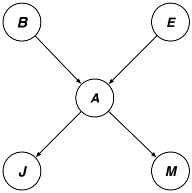

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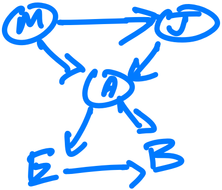
$P(M, J, A, E, B)$ = P(M) P(J|M) P(A|J M) P(E|A J M) P(B|E A J M)

P(J) ~~✗~~ P(J|M)

P(E|A)

P(B|A, E)

P(B|A)



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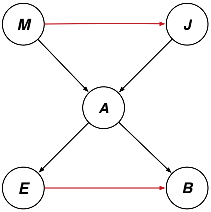

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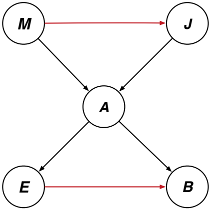

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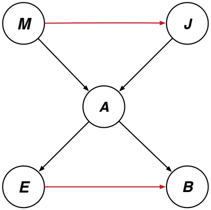

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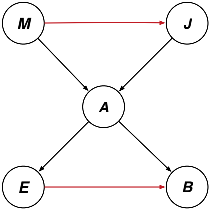

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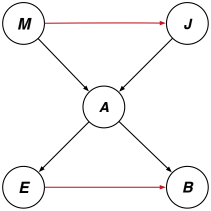

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These CPTs may be more difficult to assess.

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That’s all folks!