CSE 150A-250A AI: Probabilistic Methods

Lecture 4

Fall 2025

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Slides adapted from previous versions of the course (Prof. Lawrence, Prof. Alvarado, Prof. Berg-Kirkpatrick)

Agenda

Review

Conditional probability tables

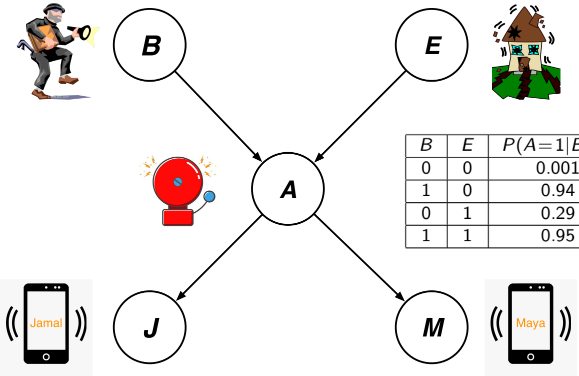
d-separation and examples

Review

Alarm example

$P(B=1) = 0.001$

$P(E=1) = 0.002$



B	E	$P(A=1\|B,E)$
0	0	0.001
1	0	0.94
0	1	0.29
1	1	0.95

A	$P(J=1\|A)$
0	0.05
1	0.9

A	$P(M=1\|A)$
0	0.01
1	0.7

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1. Nodes represent random variables.
2. Edges represent (direct) dependencies.
3. Conditional probability tables (CPTs) describe how each node depends on its parents.

BN = DAG + CPTs

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In words: Each variable is conditionally independent of its non-descendants given it's parents.

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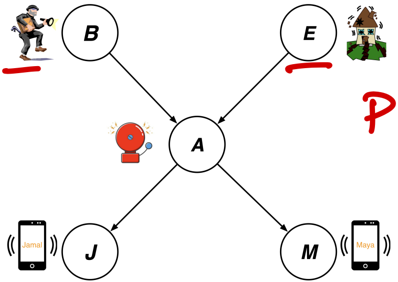

$$P(B,E,A,J,M) = P(B)\cdot P(E) \, P(A|B,E)\cdot P(J|A) \, P(M|A)$$

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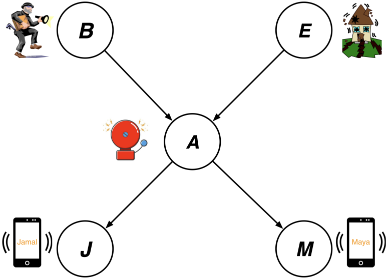

$$\begin{aligned} P(E) &= P(E|B) \\ P(J|A) &= P(J|A,B,E) \\ P(M|A) &= P(M|A,B,E,J) \end{aligned}$$

Marginal and conditional independence in DAGs

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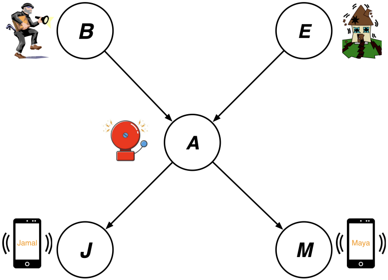

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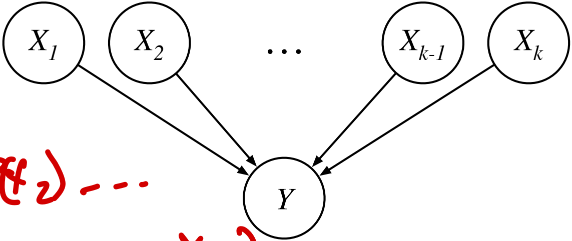
$$

These are true no matter what CPTs are attached to the nodes in the DAG.

Conditional probability tables

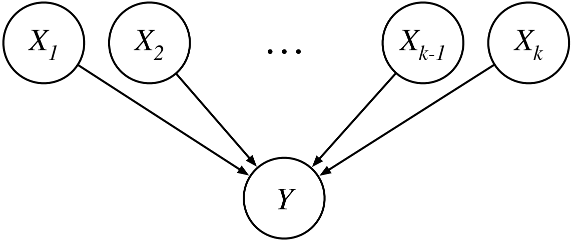
Representing CPTs



$P(x_1)P(x_2)\ldots$

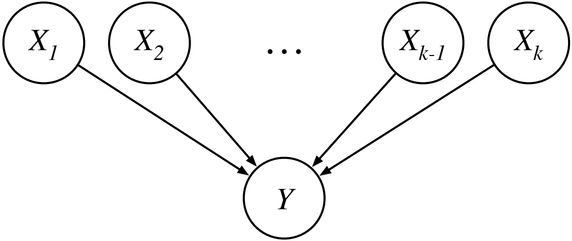
$P(y \mid x_1 \ldots x_k)$

Representing CPTs



- How to represent $P(Y|X_1, X_2, \ldots, X_k)$?

Representing CPTs



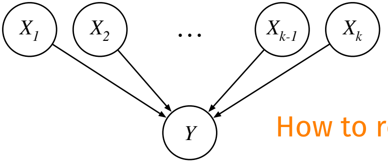
- **How to represent $P(Y|X_1, X_2, \ldots, X_k)$?**

- **Simplest case:**

  Suppose $X_i \in \{0, 1\}$, $Y \in \{0, 1\}$ are **binary** random variables.

  How to represent $P(Y = 1|X_1, X_2, \ldots, X_k)$?

Types of CPTs



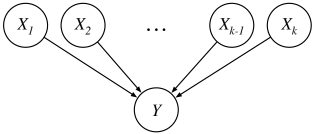
How to represent $P(Y=1|X_1, X_2, \ldots, X_k)$?

2^k

Some possibilities:

1. Tabular
2. Logical / Deterministic
3. Noisy-OR
4. Sigmoid

1. Tabular CPT



X_1	X_2	$\cdots$	X_k	$P(Y=1 \mid X_1, X_2, \ldots, X_k)$
0	0	$\cdots$	0	0.1
1	0	$\cdots$	0	0.6
0	1	$\cdots$	0	0.3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	1	$\cdots$	1	0.2

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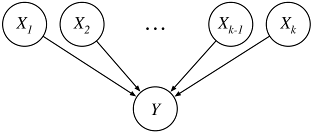

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A lookup table can exhaustively enumerate a conditional probability for every configuration of parents.

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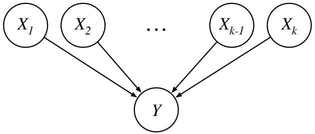

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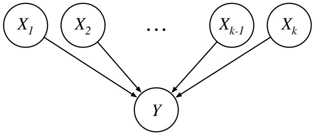

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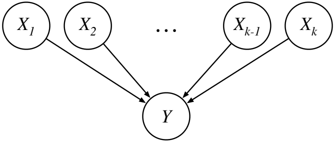
Con A table with 2^k rows is too unwieldy for large k.

2. Logical / Deterministic CPT



CPTs can also mimic the behavior of logical circuits.

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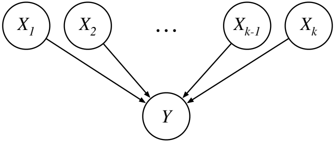

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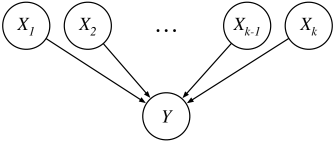

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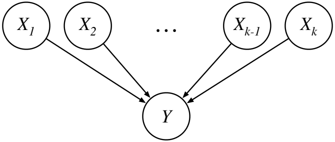

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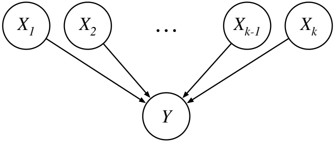

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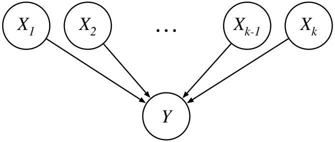

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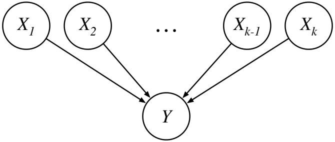

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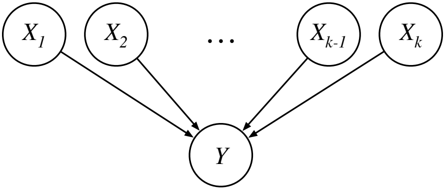
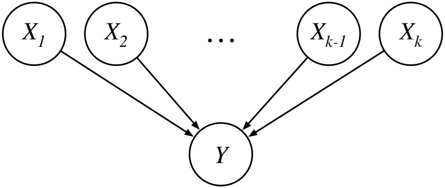
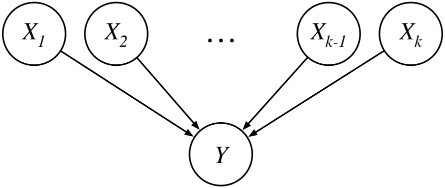


X_1
X_2
…
X_{k-1}
X_k
Y

3. Noisy-OR CPT



Use k numbers $p_i \in [0, 1]$ to parameterize all 2^k entries in the CPT:



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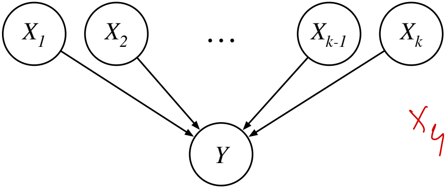
$$P(Y=0|X_1, X_2, \ldots, X_k) \quad = \quad \prod_{i=1}^{k} (1 - p_i)^{X_i}$$

$(1 - X_i)$



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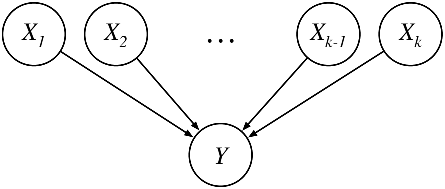

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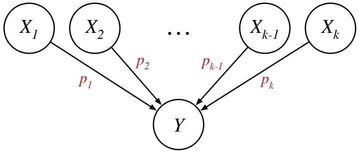
But why is this called Noisy-OR?

Noisy-OR CPT (con't)



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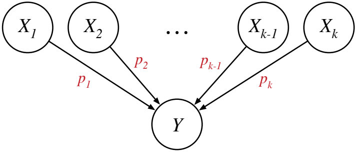

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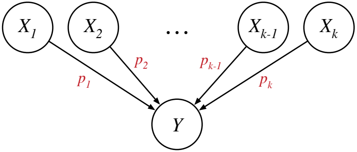

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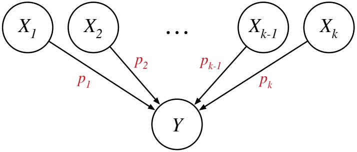

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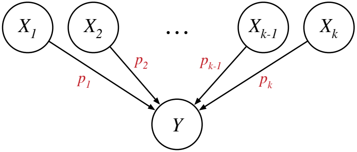

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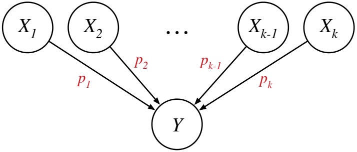

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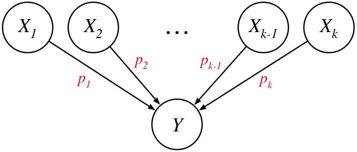

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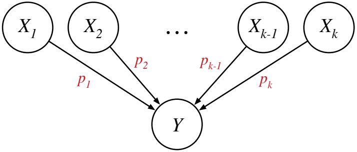

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- When exactly one parent X_j is equal to one:

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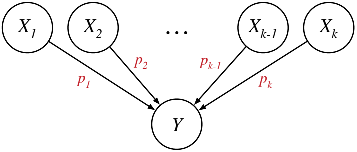

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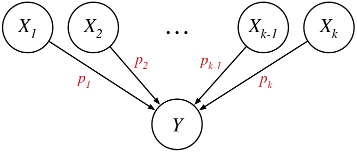
$$P(Y=1|X_1=0,\ldots,X_{j-1}=0,X_j=1,X_{j+1}=0,\ldots,X_k=0)$$



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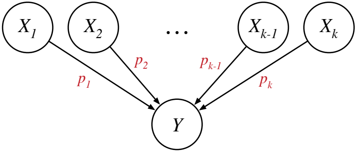
Noisy-OR CPT (con't)



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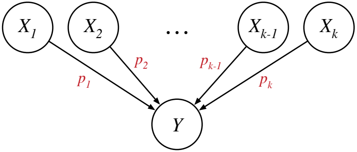
$$P(Y=1|X_1=0,X_2=0,\ldots,X_k=0) \;=\; 1-\prod_{i=1}^{k}(1-p_i)^0 \;=\; 1-\prod_{i=1}^{k}(1) \;=\; 0$$

- When exactly one parent X_j is equal to one:

$$\begin{aligned} &P(Y=1|X_1=0,\ldots,X_{j-1}=0,X_j=1,X_{j+1}=0,\ldots,X_k=0) \\ &\quad= 1-(1-p_1)^0\cdots(1-p_{j-1})^0(1-p_j)^1(1-p_{j+1})^0\cdots(1-p_k)^0 \\ &\quad= 1-(1-p_j) \end{aligned}$$

Noisy-OR CPT (con't)



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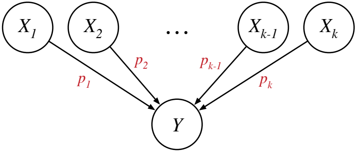

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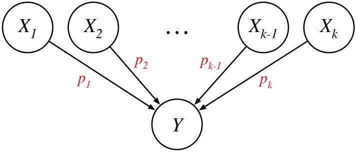

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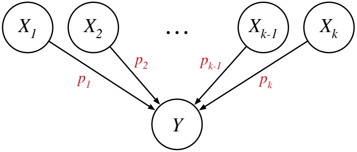

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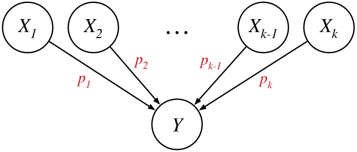

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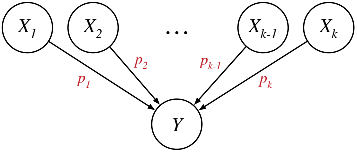

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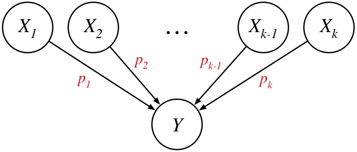

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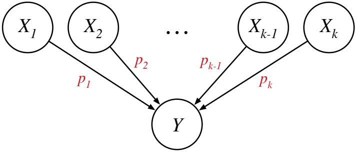

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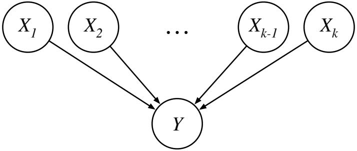
  The parents $\{X_i\}_{i=1}^{k}$ are diseases, and the child Y is a symptom. The more diseases, the more likely is the symptom.

4. Sigmoid CPT



Use k real numbers $\theta_i \in \Re$ to parameterize all 2^k entries in the CPT:

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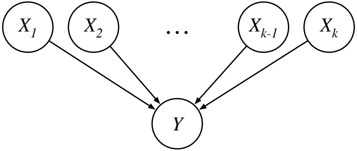

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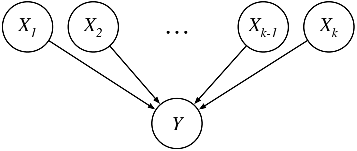

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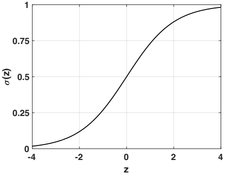
The function on the right hand side is called the **sigmoid** function:

$$\sigma(z) \;=\; \frac{1}{1+e^{-z}}$$



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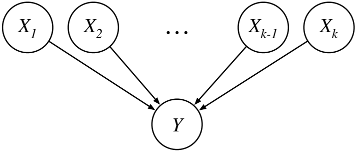

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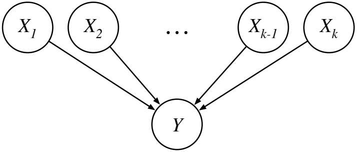

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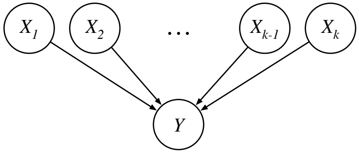

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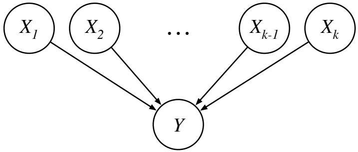

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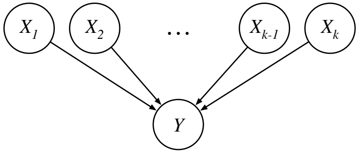

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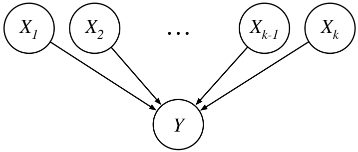

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- These effects can mix in a sigmoid CPT (unlike noisy-OR).

d-separation and examples

Conditional independence

- What we’ve already seen

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  A node X_i is conditionally independent of its non-parent ancestors given its parents:

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$X \perp\!\!\!\perp Y \mid E$

$\perp$

$$\text{When is} \quad \left\{ \begin{array}{lcl} P(X|E,Y) & = & P(X|E) \\ P(Y|E,X) & = & P(Y|E) \\ P(X,Y|E) & = & P(X|E)\,P(Y|E) \end{array} \right\} \quad ?$$

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- **Above is special case**

$X_i \perp Y \mid pa(X_i)$

$$X = \{X_i\}, \quad E = \mathrm{pa}(X_i) \quad Y = \{X_1, X_2, \ldots, X_{i-1}\} - \mathrm{pa}(X_i)$$

Base Cases

d-separation in DAGs

d-separation = direction-dependent separation

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What counts as a path, and when is it blocked?

Paths in DAGs

- Definition

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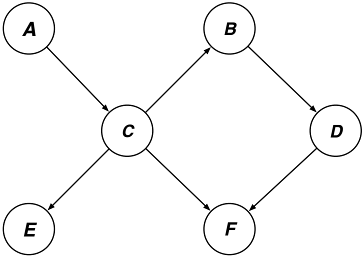

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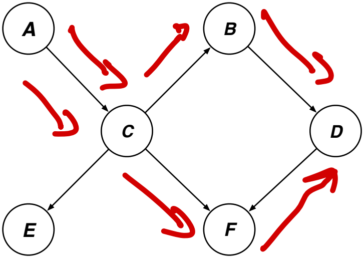

Two ? paths from A to D:

(1) $A \rightarrow C \rightarrow B \rightarrow D$

(2) $A \rightarrow C \rightarrow F \leftarrow D$

Blocked paths

- **Definition**

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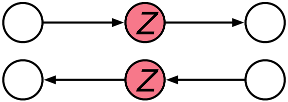

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(1) $Z \in E$ — edges **align**

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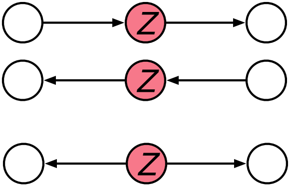

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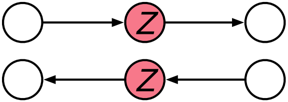
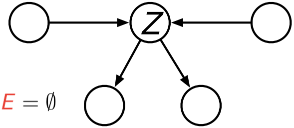


edges **align**

(2) $Z \in E$

edges **diverge**

(3) $Z \notin E$



edges **converge**

$\text{descendants}(Z) \cap E = \emptyset$

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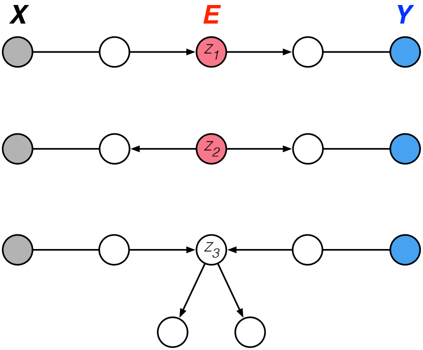

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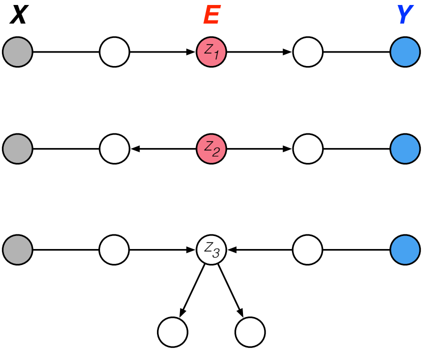

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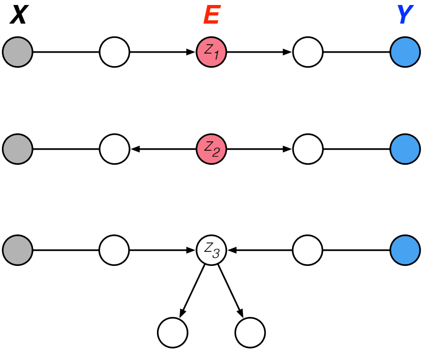

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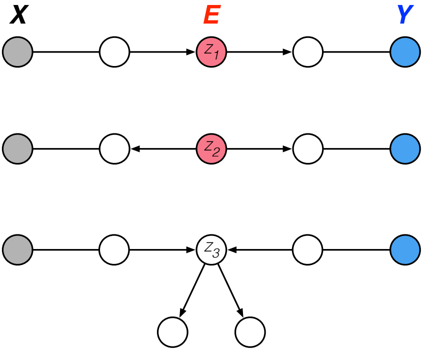

$Z_1 \in E$ is an intervening event in a causal chain

$Z_2 \in E$ is a common explanation or cause

$Z_3 \notin E, \mathrm{desc}(Z_3) \cap E = \emptyset$ is an unobserved common effect

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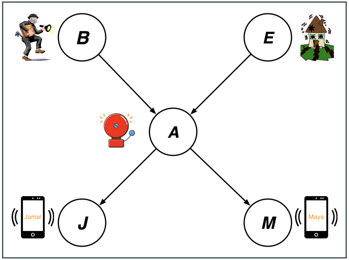
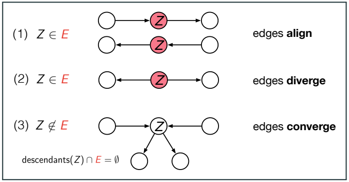
  There are efficient algorithms to test d-separation in large BNs.
  You should become skilled at these tests in simple BNs.

Alarm example

A. TRUE or B. FALSE?





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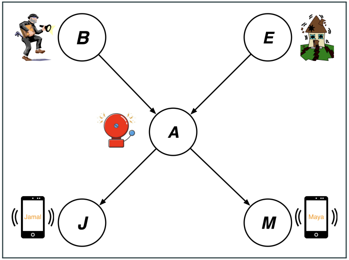
A. TRUE or B. FALSE?

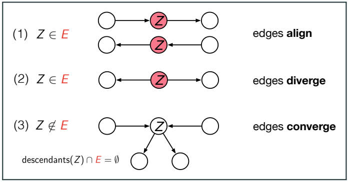
1. $P(B|A, M) \stackrel{?}{=} P(B|A)$

$X = \{B\}$

$Y = \{M\}$

$E = \{A\}$



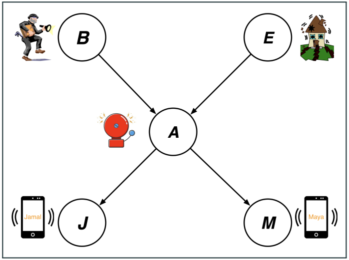


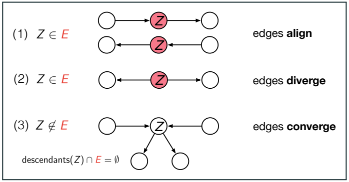
Alarm example

A. TRUE or B. FALSE?

1. $P(B|A, M) \stackrel{?}{=} P(B|A)$

   The evidence is $\{A\}$.





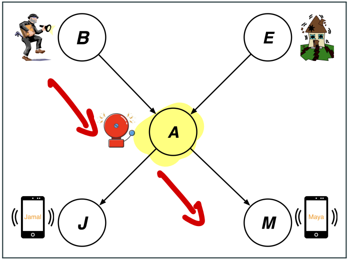
Alarm example

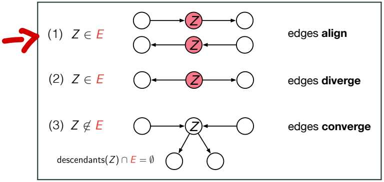
A. TRUE or B. FALSE?

1. $P(B|A, M) \stackrel{?}{=} P(B|A)$

   The evidence is $\{A\}$.

   There is one path $B \rightarrow A \rightarrow M$.



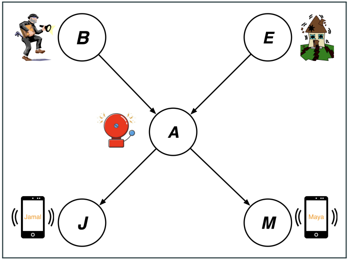


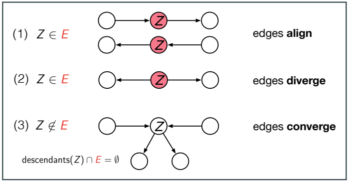
Alarm example

A. TRUE or B. FALSE?

1. $P(B|A, M) \stackrel{?}{=} P(B|A)$

   The evidence is $\{A\}$.
   There is one path $B \rightarrow A \rightarrow M$.
   Node A satisfies condition (1).



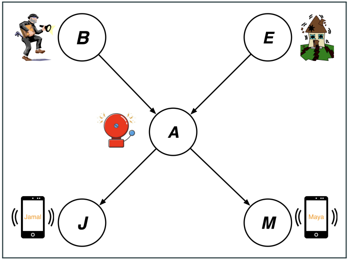


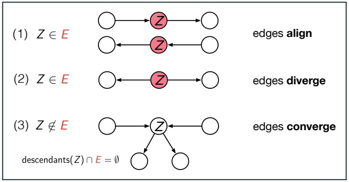
Alarm example

A. TRUE or B. FALSE?

1. $P(B|A, M) \stackrel{?}{=} P(B|A)$

   The evidence is $\{A\}$.
   There is one path $B \to A \to M$.
   Node A satisfies condition (1).
   The statement is **true**.





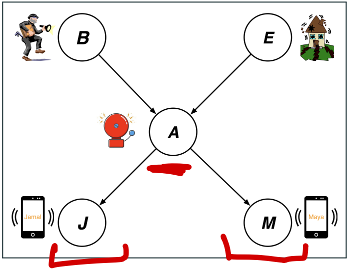
Alarm example

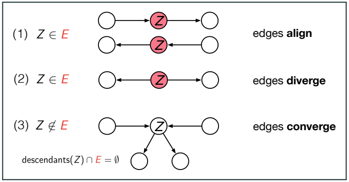
A. TRUE or B. FALSE?

1. $P(B|A, M) \stackrel{?}{=} P(B|A)$

   The evidence is $\{A\}$.
   There is one path $B \rightarrow A \rightarrow M$.
   Node A satisfies condition (1).
   The statement is **true**.

2. $P(J, M|A) \stackrel{?}{=} P(J|A)\,P(M|A)$





Alarm example

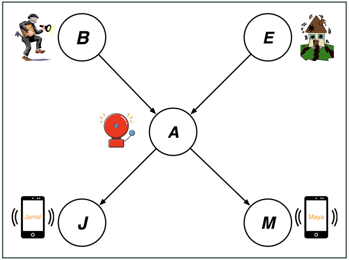
A. TRUE or B. FALSE?

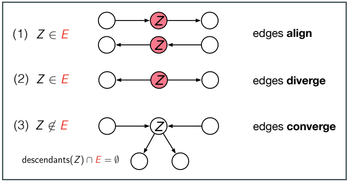
1. $P(B|A, M) \stackrel{?}{=} P(B|A)$

   The evidence is $\{A\}$.
   There is one path $B \rightarrow A \rightarrow M$.
   Node A satisfies condition (1).
   The statement is **true**.

2. $P(J, M|A) \stackrel{?}{=} P(J|A)\, P(M|A)$

   The evidence is $\{A\}$.





Alarm example

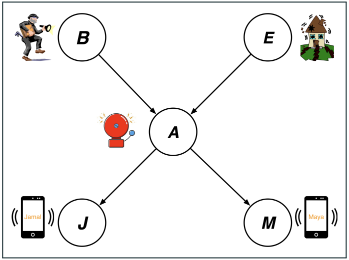
A. TRUE or B. FALSE?

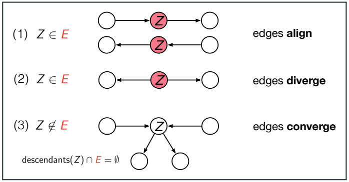
1. $P(B|A, M) \stackrel{?}{=} P(B|A)$

   The evidence is $\{A\}$.
   There is one path $B \to A \to M$.
   Node A satisfies condition (1).
   The statement is **true**.

2. $P(J, M|A) \stackrel{?}{=} P(J|A)\,P(M|A)$

   The evidence is $\{A\}$.
   There is one path $J \leftarrow A \to M$.





Alarm example

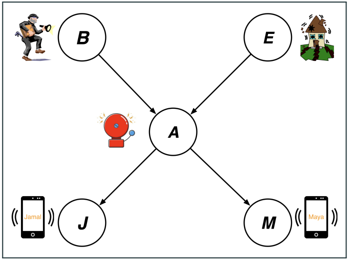
A. TRUE or B. FALSE?

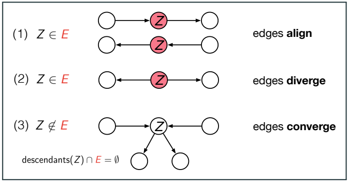
1. $P(B|A, M) \stackrel{?}{=} P(B|A)$

   The evidence is $\{A\}$.
   There is one path $B \rightarrow A \rightarrow M$.
   Node A satisfies condition (1).
   The statement is **true**.

2. $P(J, M|A) \stackrel{?}{=} P(J|A)\,P(M|A)$

   The evidence is $\{A\}$.
   There is one path $J \leftarrow A \rightarrow M$.
   Node A satisfies condition (2).





Alarm example

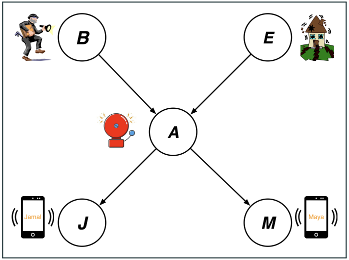
A. TRUE or B. FALSE?

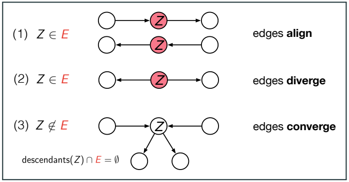
1. $P(B|A, M) \stackrel{?}{=} P(B|A)$

   The evidence is $\{A\}$.
   There is one path $B \rightarrow A \rightarrow M$.
   Node A satisfies condition (1).
   The statement is **true**.

2. $P(J, M|A) \stackrel{?}{=} P(J|A)\,P(M|A)$

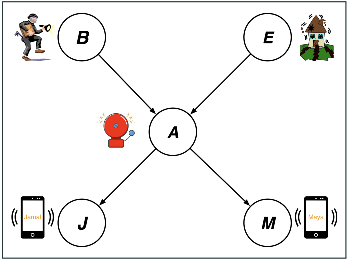
   The evidence is $\{A\}$.
   There is one path $J \leftarrow A \rightarrow M$.
   Node A satisfies condition (2).
   The statement is **true**.

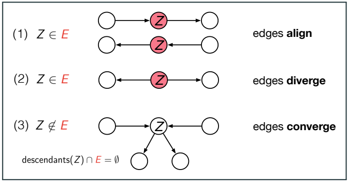




Alarm example (con't)

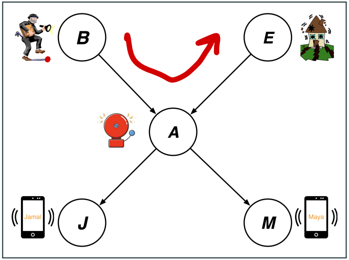
A. TRUE or B. FALSE?

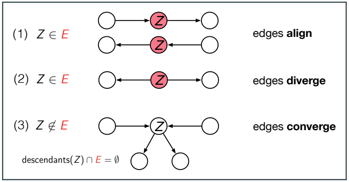




A. TRUE or B. FALSE?

3. $P(B) \stackrel{?}{=} P(B|E)$



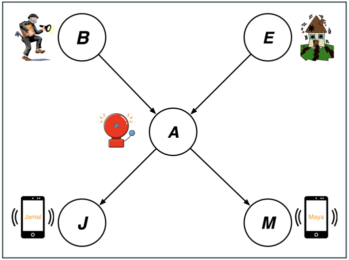


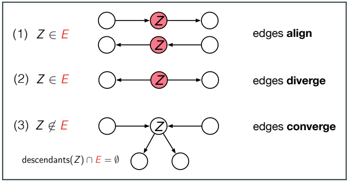
Alarm example (con't)

A. TRUE or B. FALSE?

3. $P(B) \stackrel{?}{=} P(B|E)$

   The evidence is $\{\}$.





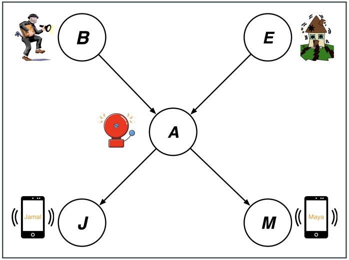
Alarm example (con't)

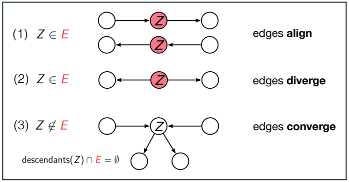
A. TRUE or B. FALSE?

3. $P(B) \stackrel{?}{=} P(B|E)$

   The evidence is $\{\}$.

   There is one path $B \rightarrow A \leftarrow E$.



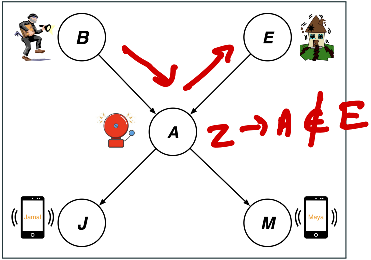


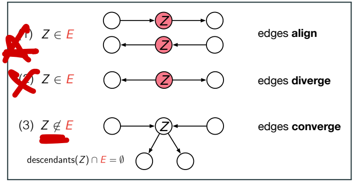
Alarm example (con't)

A. TRUE or B. FALSE?

3. $P(B) \stackrel{?}{=} P(B|E)$

   The evidence is $\{\}$.
   There is one path $B \rightarrow A \leftarrow E$.
   Node A satisfies condition (3).



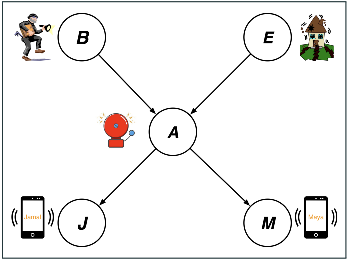


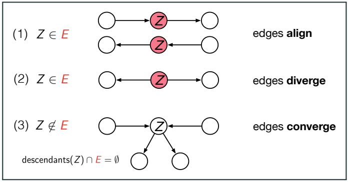
Alarm example (con't)

A. TRUE or B. FALSE?

3. $P(B) \stackrel{?}{=} P(B|E)$

   The evidence is $\{\}$.
   There is one path $B \rightarrow A \leftarrow E$.
   Node A satisfies condition (3).
   The statement is **true**.





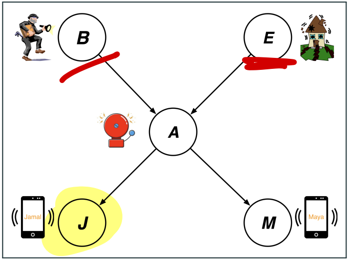
Alarm example (con't)

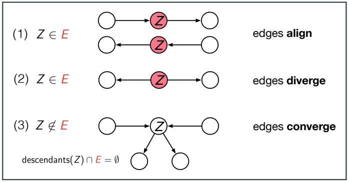
A. TRUE or B. FALSE?

3. $P(B) \stackrel{?}{=} P(B|E)$

   The evidence is $\{\}$.
   There is one path $B \to A \leftarrow E$.
   Node A satisfies condition (3).
   The statement is **true**.

4. $P(B|M) \stackrel{?}{=} P(B|M, E)$





Alarm example (con't)

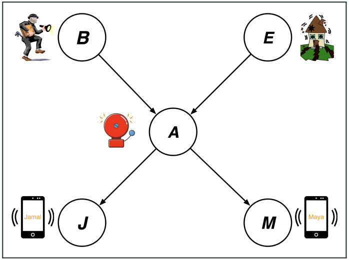
A. TRUE or B. FALSE?

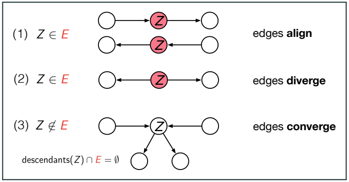
3. $P(B) \stackrel{?}{=} P(B|E)$

   The evidence is $\{\}$.
   There is one path $B \rightarrow A \leftarrow E$.
   Node A satisfies condition (3).
   The statement is **true**.

4. $P(B|M) \stackrel{?}{=} P(B|M, E)$

   The evidence is $\{M\}$.





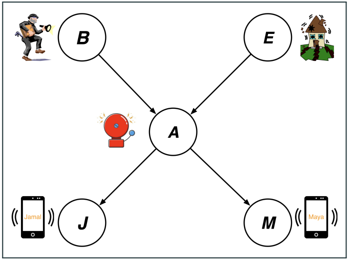
A. TRUE or B. FALSE?

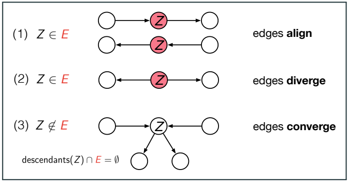
3. $P(B) \stackrel{?}{=} P(B|E)$

   The evidence is $\{\}$.
   There is one path $B \rightarrow A \leftarrow E$.
   Node A satisfies condition (3).
   The statement is **true**.

4. $P(B|M) \stackrel{?}{=} P(B|M, E)$

   The evidence is $\{M\}$.
   There is one path $B \rightarrow A \leftarrow E$.





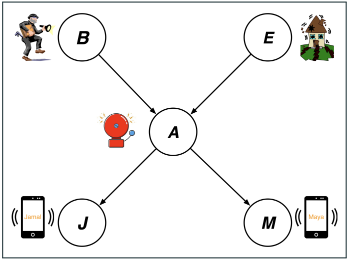
A. TRUE or B. FALSE?

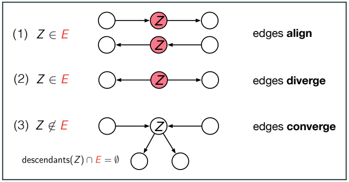
3. $P(B) \stackrel{?}{=} P(B|E)$

   The evidence is $\{\}$.
   There is one path $B \rightarrow A \leftarrow E$.
   Node A satisfies condition (3).
   The statement is **true**.

4. $P(B|M) \stackrel{?}{=} P(B|M, E)$

   The evidence is $\{M\}$.
   There is one path $B \rightarrow A \leftarrow E$.
   Note that $M \in \mathrm{desc}(A)$.





Alarm example (con't)

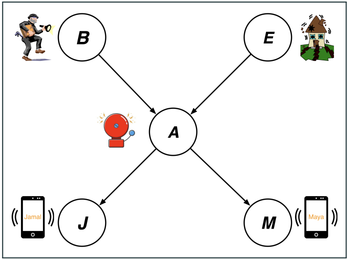
A. TRUE or B. FALSE?

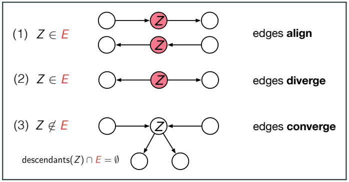
3. $P(B) \stackrel{?}{=} P(B|E)$

   The evidence is $\{\}$.
   There is one path $B \rightarrow A \leftarrow E$.
   Node A satisfies condition (3).
   The statement is **true**.

4. $P(B|M) \stackrel{?}{=} P(B|M, E)$

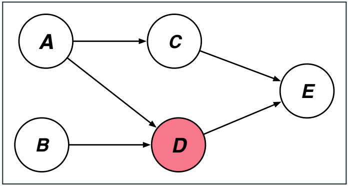
   The evidence is $\{M\}$.
   There is one path $B \rightarrow A \leftarrow E$.
   Note that $M \in \mathrm{desc}(A)$.
   The statement is **false**.

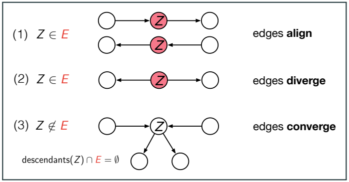




Loopy example

A. TRUE or B. FALSE?

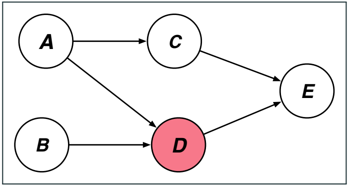


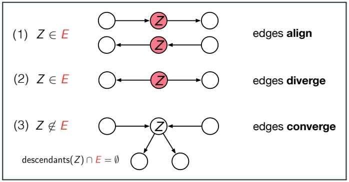


Loopy example

A. TRUE or B. FALSE?

5. $P(B|D, E) \stackrel{?}{=} P(B|D)$



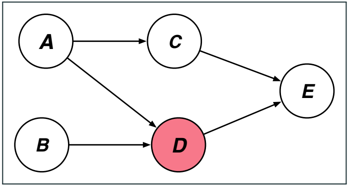


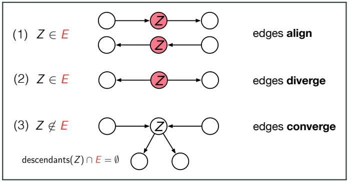
Loopy example

A. TRUE or B. FALSE?

5. $P(B|D, E) \stackrel{?}{=} P(B|D)$

   The evidence is $\{D\}$.





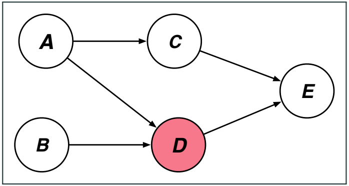
Loopy example

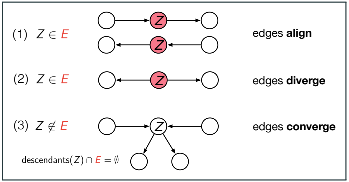
A. TRUE or B. FALSE?

5. $P(B|D, E) \stackrel{?}{=} P(B|D)$

   The evidence is $\{D\}$.

   There are two paths from B to E.





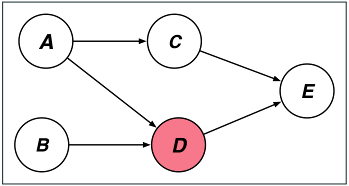
Loopy example

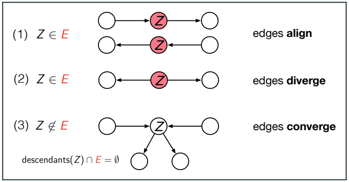
A. TRUE or B. FALSE?

5. $P(B|D,E) \stackrel{?}{=} P(B|D)$

   The evidence is $\{D\}$.
   There are two paths from B to E.

   Path $B \rightarrow D \rightarrow E$
   is blocked by node D,
   satisfying condition (1).





Loopy example

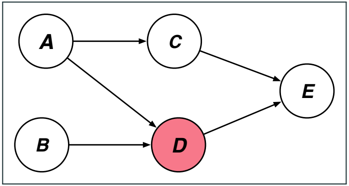
A. TRUE or B. FALSE?

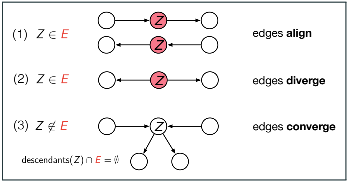
5. $P(B|D, E) \stackrel{?}{=} P(B|D)$

   The evidence is $\{D\}$.
   There are two paths from B to E.

   Path $B \rightarrow D \rightarrow E$
   is blocked by node D,
   satisfying condition (1).

   Path $B \rightarrow D \leftarrow A \rightarrow C \rightarrow E$
   is not blocked by any node.





A. TRUE or B. FALSE?

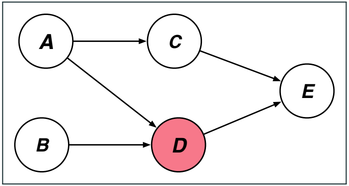
5. $P(B|D,E) \stackrel{?}{=} P(B|D)$

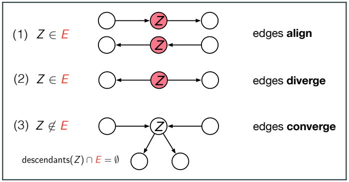
The evidence is $\{D\}$.
There are two paths from B to E.

Path $B \to D \to E$
is blocked by node D,
satisfying condition (1).

Path $B \to D \leftarrow A \to C \to E$
is not blocked by any node.

The statement is **false**.





Markov Blanket

A Markov Blanket B_X of node X consists of **parents** of X, **children** of X and **"spouses"** (other parents of children of X, but not X) of X.

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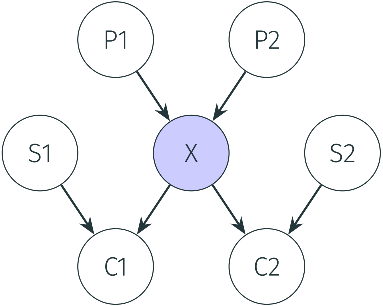

Every variable is conditionally independent of any other variable given it's Markov Blanket.

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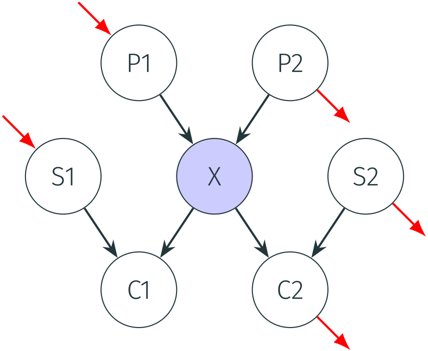

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That's all folks!