Discussion for hw1

1.1 Conditioning on background evidence (5 pts)

It is often useful to consider the impact of specific events in the context of general background evidence, rather than in the absence of information.

(a) Denoting such evidence by E, prove the conditionalized version of the product rule:

$$P(X, Y|E) \ = \ P(X|Y, E)P(Y|E).$$

Use the standard product rule: P(X,Y,E) = P(X,Y|E)P(E)

(b) Also, prove the conditionalized version of Bayes rule:

$$P(X|Y, E) \ = \ \frac{P(Y|X, E)P(X|E)}{P(Y|E)}.$$

- Like how we prove the standard Bayes rule, we need to make use of product rule.
- Apply (a) twice, switching X and Y

(c) Also, prove the conditionalized version of marginalization:

$$P(X|E) \ = \ \sum_{y} P(X, Y = y|E).$$

Use the general product rule: P(X,Y|E) = $\frac{P(X,YE)}{P(E)}$, Then we marginalize on Y

1.2 Conditional independence (5 pts)

Show that the following three statements about random variables X, Y, and E are equivalent:

$$
\begin{aligned}
(1) \quad P(X, Y|E) &= P(X|E)P(Y|E) \\
(2) \quad P(X|Y, E) &= P(X|E) \\
(3) \quad P(Y|X, E) &= P(Y|E)
\end{aligned}
$$

In other words, show that (1) implies (2) & (3), that (2) implies (1) & (3), and that (3) implies (1) & (2). You should become fluent with all these ways of expressing that X is conditionally independent of Y given E.

- Prove equivalence : a. cyclical implication:

  (1) implies (2), (2) implies (3) and then (3) implies (1)

  b. pairwise equivalence:

  (1) ⇔ (2) , (2) ⇔ (3)

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- Example: (1) ⇒ (2)
  Reuse what we have proven in 1.1: **conditionalized version product rule:**

1.3 Creative writing (5 pts)

This problem does not involve any calculations: simply attach events to the binary random variables X, Y, and Z that are consistent with the following patterns of commonsense reasoning. You may use different events for the different parts of the problem. Also, please be creative: do not use the same events (e.g., burglaries, earthquakes, alarms) that were considered in lecture.

(a) Cumulative evidence:

$$P(X=1) < P(X=1|Y=1) < P(X=1|Y=1, Z=1)$$

Consider a single effect X with multiple causes Y , Z. For example: here I use the same events considered in lecture:

P(alarms = 1) < P(alarms = 1| earthquakes = 1) < P(alarms = 1| earthquakes = 1,burglaries = 1)

(b) Explaining away:

$$P(X=1|Y=1) > P(X=1),$$
$$P(X=1|Y=1, Z=1) < P(X=1|Y=1)$$

Consider a single effect Y with independent causes X, Z.

X⟶Y⟵Z

e.g., B→A←E.

- If you hear the alarm (A=1), that raises the probability that **at least one** cause happened. P(B=1|A=1) > P(B=1)

- If you also observe an earthquake (E=1), that provides an alternate explanation for the alarm, so the probability of burglary drops.
  P(B=1|A=1,E=1) > P(B=1|A=1)

(c) Conditional independence:

$$
\begin{aligned}
P(X=1, Y=1) &\neq P(X=1)P(Y=1) \\
P(X=1, Y=1|Z=1) &= P(X=1|Z=1)P(Y=1|Z=1)
\end{aligned}
$$

In a **common cause** structure:

$$X \longleftarrow Z \longrightarrow Y$$

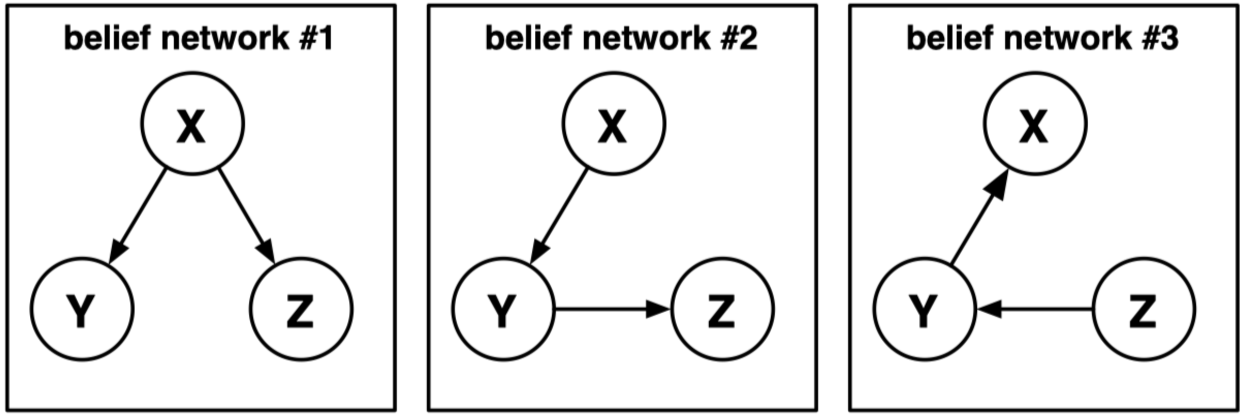
Z: earthquake
X: alarm
Y: traffic jam

- If you hear the alarm (X=1), you increase your belief that there was an earthquake (Z=1). If an earthquake is more likely, then a traffic jam (Y=1) is also more likely. So X and Y are dependent in the marginal distribution.

- If you already know whether an earthquake happened (Z), then the alarm and the traffic jam become independent of each other.

1.4 Compare and contrast (5 pts)

Consider the different belief networks (BNs) shown below for the discrete random variables X, Y, and Z.



(a) Does the first belief network imply a statement of marginal or conditional independence that is not implied by the second? If yes, provide an example.

(b) Does the second belief network imply a statement of marginal or conditional independence that is not implied by the third? If yes, provide an example.

(c) Does the third belief network imply a statement of marginal or conditional independence that is not implied by the first? If yes, provide an example.

- Write down the marginal or conditional independence implied by these BN
- For example, for BN #1, we have Y and Z are independent given X

$$P(Y, Z| X) = P(Y| X) P(Z|X)$$

- Then we check if we can infer this independence from other BNs.